Example: Fruit in Boxes

- Random variables:
  - Fruit: \( F = \{ \text{apple}(a), \text{orange}(o) \} \)
  - Box: \( B = \{ \text{red}(r), \text{blue}(b) \} \)
- Prior Box distribution:
  
  \[
  p(B = r) = \frac{4}{10} \\
  p(B = b) = \frac{6}{10}
  \]
- Conditional probabilities of Fruit given Box
  
  \[
  p(F = a \mid B = b) = \frac{3}{4} \\
  p(F = a \mid B = r) = \frac{6}{8} = \frac{3}{4} \\
  p(F = o \mid B = b) = \frac{1}{4} \\
  p(F = o \mid B = r) = \frac{5}{8} = \frac{1}{4} \]
- Marginal Fruit distributions:
  
  \[
  p(F = a) = \sum_B p(F = a, B) = \sum_B p(F = a \mid B) \cdot p(B) = \frac{3}{4} \cdot \frac{6}{10} + \frac{1}{4} \cdot \frac{4}{10} = \frac{11}{20} \\
  p(F = o) = \frac{3}{4} \cdot \frac{4}{10} + \frac{1}{4} \cdot \frac{6}{10} = \frac{9}{20} = \left( 1 - \frac{11}{20} \right) = \frac{9}{20}
  \]
Example: Fruit in Boxes

- Prior: \( p( B = r ) = 4/10 \) \& \( p( B = b ) = 6/10 \)
- Marginal: \( p( F=a ) = 11/20 \) \& \( p( F = o) = 9/20 \)
- Posterior probability of Box color given observed fruit
  
  \[
  p(B = r | F = o) = \frac{p(F = o | B = r) \cdot p(B = r)}{p(F = o)}
  \]

  \[
  = \frac{6/9 \cdot 4/10}{9/20} = \frac{6}{9} = \frac{2}{3} \approx 0.667
  \]

- Prior probability of red box:
  \( p(B = r) = 4/10 \) \( < p(B = r | F = o) \)
- After observing an orange the probability of observing a red box is now larger than observing a blue box!

Figure: coloured boxes containing apples and oranges (Bishop 1.9)
Independent Random Variables

Two random variables $X$ and $Y$ are independent iff measuring $X$ gives no information on $Y$, and vice versa.

- Formally: $X$ and $Y$ are called independent if

$$p(x, y) = p(x) \cdot p(y)$$

- Equivalent to

$$p(x|y) = \frac{p(x,y)}{p(y)} = \frac{p(x)p(y)}{p(y)} = p(x)$$

- Example:

$$p(F|B) = p(F) = \frac{1}{2}$$

$$p(B|F) = p(B)$$